

# PRIMITIVE INVERSE SEMIGROUPS OF LEFT I-QUOTIENTS

N. GHRODA

**ABSTRACT.** A subsemigroup  $S$  of an inverse semigroup  $Q$  is a left I-order in  $Q$ , if every element in  $Q$  can be written as  $a^{-1}b$  where  $a, b \in S$  and  $a^{-1}$  is the inverse of  $a$  in the sense of inverse semigroup theory. We study a characterisation of semigroups which have a primitive inverse semigroup of left I-quotients.

## 1. INTRODUCTION

Clifford [2] showed that from any right cancellative monoid  $S$  with (LC) Condition, there is a bisimple inverse monoid  $Q$  such that  $Q = S^{-1}S$ , that is, every element  $q$  in  $Q$  can be written as  $a^{-1}b$  where  $a, b \in S$ . By saying that a semigroup  $S$  has the (LC) Condition we mean for any  $a, b \in S$  there is an element  $c \in S$  such that  $Sa \cap Sb = Sc$ . In [9] the authors have extended Clifford's work to a left ample semigroup with (LC) where they introduced the following definition of left I-order in inverse semigroups.

Let  $Q$  be an inverse semigroup. A subsemigroup  $S$  of  $Q$  is a *left I-order* in  $Q$  and  $Q$  is a *semigroup of left I-quotient* of  $S$ , if every element in  $Q$  can be written as  $a^{-1}b$  where  $a, b \in S$  and  $a^{-1}$  is the inverse of  $a$  in the sense of inverse semigroup theory. *Right I-order* and *semigroup of right I-quotients* is defined dually. If  $S$  is both a left and a right I-order in an inverse semigroup  $Q$ , we say that  $S$  is an *I-order* in  $Q$  and  $Q$  is a *semigroup of I-quotients* of  $S$ . This notion extends the classical notion of left order in an inverse semigroup  $Q$ , due to Fountain and Petrich [3]. We say that a subsemigroup  $S$  of a semigroup  $Q$  is a *left order* in  $Q$  or  $Q$  is a *semigroup of left quotients* of  $S$  if every element of  $Q$  can be written as  $a^\#b$  where  $a, b \in S$  and  $a^\#$  is the inverse of  $a$  in a subgroup of  $Q$  and if, in addition, every square-cancellable element (an element  $a$  of a semigroup  $S$  is square-cancellable if  $a\mathcal{H}^*a^2$ ) lies in a subgroup of  $Q$ .

Clearly if  $S$  has an inverse semigroup of left quotients  $Q$ , then  $Q$  is also a semigroup of left I-quotients, but the converse is not true as will see by an example. *Right order* and *semigroup of right quotients* are defined dually. If  $S$  is both a left and right order in  $Q$ , then  $S$  is an *order* in  $Q$  and  $Q$  is a *semigroup of quotients*

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of  $S$ .

In this article we focus on studying left I-orders in primitive inverse semigroups.

In Section 2 we begin by investigating some properties of semigroups which are left I-orders in primitive inverse semigroups. The next section is devoted to the proof of Theorem 3.1 which characterizes those semigroups which have a primitive inverse semigroup of left I-quotients. We then specialise our result to left I-orders in Brandt semigroups a result which may be regarded as a generalisation of the main theorem in [7], which characterised left orders in Brandt semigroups. The statement of an alternative description of left I-orders in Brandt semigroups has been privately communicated by Cegarra [1].

In Section 4 we show that a primitive inverse semigroup of left I-quotients is unique up to isomorphism. Section 5 then concentrates on I-orders (two-sided case) in primitive inverse semigroups. In the final Section we give characterizations of adequate (ample) semigroups having primitive inverse semigroups of left I-quotients.

## 2. PRELIMINARIES

Throughout this article, we shall follow the terminologies and notation of [2]. The set of non-zero elements of a semigroup  $S$  will be denoted by  $S^*$ .

The relations  $\mathcal{R}^*$ ,  $\mathcal{L}^*$  and  $\mathcal{H}^*$  play a significant role in this article. It is well known that the relation  $\mathcal{R}^*$  is defined on a semigroup  $S$  by the rule that  $a \mathcal{R}^* b$  in  $S$  if  $a \mathcal{R} b$  in some oversemigroup  $T$  of  $S$ , and this equivalent to  $a \mathcal{R}^* b$  if and only if

$$xa = ya \quad \text{if and only if} \quad xb = yb$$

for all  $x, y \in S^1$ . The relation  $\mathcal{L}^*$  is defined dually and  $\mathcal{H}^* = \mathcal{R}^* \cap \mathcal{L}^*$ . It is clear that  $\mathcal{R} \subseteq \mathcal{R}^*$  and  $\mathcal{L} \subseteq \mathcal{L}^*$  where  $\mathcal{R}$  and  $\mathcal{L}$  are the usual Green's relations.

We recall that a semigroup  $S$  with zero is defined to be *categorical at 0* if whenever  $a, b, c \in S$  are such that  $ab \neq 0$  and  $bc \neq 0$ , then  $abc \neq 0$ . We say that,  $S$  is *0-cancellative* if  $b = c$  follows from  $ab = ac \neq 0$  and from  $ba = ca \neq 0$ .

An inverse semigroup  $S$  with zero is a *primitive inverse* semigroup if all its nonzero idempotents are primitive, where an idempotent  $e$  of  $S$  is called *primitive* if  $e \neq 0$  and  $f \leq e$  implies  $f = 0$  or  $e = f$ . We will use the following facts about primitive inverse semigroups heavily through this article.

**Lemma 2.1.** [5] *Let  $Q$  be a primitive inverse semigroup.*

- (i)  *$Q$  is categorical at 0.*
- (ii) *If  $e, f \in E^*(Q)$ , then  $ef \neq 0$  implies  $e = f$ .*

(iii) If  $e \in E^*$  and  $s \in Q^*$ , then

$$es \neq 0 \text{ implies } es = s \text{ and } se \neq 0 \text{ implies } se = s.$$

(iv) If  $a, s \in Q^*$  and  $as = a$ , then  $s = a^{-1}a$ . Dually, if  $sa = a$ , then  $s = aa^{-1}$ .

(v) If  $ab \neq 0$ , then  $a^{-1}a = b^{-1}b$ .

From the above lemma we can notice easily that a primitive inverse semigroup is 0-cancellative.

To investigate the properties of a semigroup  $S$  which is a left I-order in a primitive inverse semigroup  $Q$  we need the relations  $\lambda$ ,  $\rho$  and  $\tau$  which are introduced in [6] on any semigroup with zero as follows:

$$a \lambda b \text{ if and only if } a = b = 0 \text{ or } Sa \cap Sb \neq 0$$

$$a \rho b \text{ if and only if } a = b = 0 \text{ or } aS \cap bS \neq 0$$

$$\tau = \rho \cap \lambda.$$

**Lemma 2.2.** *Let  $S$  be any semigroup with zero.*

(i) *If  $a \mathcal{R}^* b$  where  $a, b \neq 0$ , then*

$$xa \neq 0 \text{ if and only if } xb \neq 0.$$

(ii) *If  $S$  is categorical at 0 and 0-cancellative, then*

$$x, xa \neq 0 \text{ implies that } x \mathcal{R}^* xa,$$

*for any  $x, a \in S$ .*

*Proof.* (i) Clear.

(ii) Let  $x, a \in S$  with  $xa \neq 0$ . If  $u, v \in S^1$  and  $ux = vx$ , then clearly  $uxa = vxa$ .

Conversely, if  $uxa = vxa \neq 0$ , then by 0-cancellativity,  $ux = vx \neq 0$ . On the other hand, if  $uxa = vxa = 0$ , then by categoricity at 0,  $ux = vx = 0$  (note that in this case,  $u, v \neq 1$ ).  $\square$

**Definition 2.3.** Let  $S$  be a subsemigroup of an inverse semigroup  $Q$ . Then  $S$  is a *straight* left I-order in  $Q$  if, every  $q \in Q$  can be write as  $q = a^{-1}b$ , where  $a, b \in S$  and  $a$  and  $b$  are  $\mathcal{R}$ -related in  $Q$ .

In the next lemma we introduce some properties of a semigroup which has a primitive inverse semigroup of left I-quotients.

We made the convention that if  $S$  is a left I-order in  $Q$ , then  $\mathcal{R}$  and  $\mathcal{L}$  will be relations on  $Q$  and  $\mathcal{R}^*, \mathcal{L}^*, \lambda, \rho$  and  $\tau$  will refer to  $S$ .

**Proposition 2.4.** *Let  $S$  be a subsemigroup of a primitive inverse semigroup  $Q$ . If  $S$  is left I-order in  $Q$ , then:*

1)  *$S$  contains the 0 element of  $Q$ ;*

- 2)  $\mathcal{L} \cap (S \times S) = \lambda$ ;
- 3)  $S$  is a straight left  $I$ -order in  $Q$ ;
- 4)  $Sa \neq 0$  for all  $a \in S^*$ ;
- 5)  $\mathcal{R} \cap (S \times S) = \mathcal{R}^*$ ;
- 6)  $\rho \subseteq \mathcal{R}^*$ .

*Proof.* 1) If  $S \subseteq H_e$  for some  $0 \neq e \in E(Q)$ , then  $0 = a^{-1}b \in H_e$  which is a contradiction, so  $S \not\subseteq H_e$ . Thus  $0 \in S$  or there exist  $(i, a, j) \in S, i \neq j$ . Then  $(i, a, j)(i, a, j) = 0$ , so that  $0 \in S$ .

2) If  $a \lambda b$ , then  $a = b = 0$  and certainly  $a \mathcal{L} b$  in  $Q$ , or  $xa = yb \neq 0$  for some  $x, y \in S$ . In the latter case,  $a = x^{-1}yb, b = y^{-1}xa$ , so that  $a \mathcal{L} b$  in  $Q$ . Conversely, if  $a \mathcal{L} b$  in  $Q$ , then either  $a = b = 0$ , or  $a \neq 0$  and  $a = x^{-1}yb$  for some  $x^{-1}y \in Q$  where  $x, y \in S$ . Then  $xa = yb \neq 0$ . Hence  $a \lambda b$  in  $S$ . It is worth pointing out that in this case  $\lambda$  is transitive. Moreover, it is an equivalence, and  $\{0\}$  is a  $\lambda$ -class.

3) Suppose that  $0 \neq q \in Q$ , then  $q = a^{-1}b$  for some  $a, b \in S$ . Since  $a^{-1}b \neq 0$ , Lemma 2.1 gives  $aa^{-1} = bb^{-1}$  so that  $a \mathcal{R} b$  in  $Q$ . By categoricity at 0 in  $Q$  and Lemma 2.1,  $0 \neq aa^{-1}b = b$  and  $0 \neq a^{-1}bb^{-1} = a^{-1}$ , then  $aa^{-1} = aa^{-1}bb^{-1} = bb^{-1}$  in  $Q$ . Thus  $a \mathcal{R} b$  in  $Q$ .

4) Let  $a = x^{-1}y \neq 0$  for some  $x, y \in S$ , where  $x \mathcal{R} y$  in  $Q$ . By categoricity at 0 and Lemma 2.1 we have  $xa = y \neq 0$ . So  $Sa \neq 0$ .

5) It is clear that  $\mathcal{R} \cap (S \times S) \subseteq \mathcal{R}^*$ . To show that  $\mathcal{R}^* \subseteq \mathcal{R} \cap (S \times S)$ . Let  $a \mathcal{R}^* b$  in  $S$ ; from (4) there exist  $y$  in  $S$  such that  $ya \neq 0$ , by Lemma 2.2  $yb \neq 0$ . Then by Lemma 2.1 we have  $aa^{-1} = y^{-1}y = bb^{-1}$  and we get  $a \mathcal{R} b$  in  $Q$ .

6) Suppose that  $a \rho b$  in  $S$ , then  $a = b = 0$  and  $a \mathcal{R} b$  in  $Q$ , or  $ax = by \neq 0$  for some  $x, y \in S$ . Then  $b = axy^{-1}, a = byx^{-1}$ , so that  $a \mathcal{R} b$  in  $Q$ . By (5)  $a \mathcal{R}^* b$  in  $S$ .

□

By Lemma 2.1 and Proposition 2.4 the following corollaries are clear.

**Corollary 2.5.** *Let  $S$  be a left  $I$ -order in a primitive inverse semigroup  $Q$ . If  $a^{-1}b \neq 0$ , then  $a \mathcal{R} b$ . Consequently,  $a^{-1} \mathcal{R} a^{-1}b \mathcal{L} b$ .*

**Corollary 2.6.** *Let  $S$  be a left  $I$ -order in a primitive inverse semigroup  $Q$ . Let  $a^{-1}b, c^{-1}d$  be non-zero elements of  $Q$  where  $a, b, c$  and  $d$  are in  $S$ . Then*

- 1)  $a^{-1}b \mathcal{R} c^{-1}d$  if and only if  $a \lambda c$ ;
- 2)  $a^{-1}b \mathcal{L} c^{-1}d$  if and only if  $b \lambda d$ .

*Proof.* 1) We have that  $a^{-1}b \mathcal{R} c^{-1}d$  if and only if  $a^{-1} \mathcal{R} c^{-1}$  if and only if  $a \mathcal{L} c$ . By Proposition 2.4 this is equivalent to  $a \lambda c$ .

2) Similar.  $\square$

### 3. THE MAIN THEOREM

The aim of this section is to prove the following theorem;

**Theorem 3.1.** *A semigroup  $S$  is a left I-order in a primitive inverse semigroup  $Q$  if and only if  $S$  satisfies the following conditions:*

- (A)  $S$  is categorical at 0;
- (B)  $S$  is 0-cancellative;
- (C)  $\lambda$  is transitive;
- (D)  $Sa \neq 0$  for all  $a \in S$ .

*Proof.* Suppose that  $Q$  exists, then  $S$  inherits Conditions (A) and (B) from  $Q$ . By Proposition 2.4 first we have that Conditions (C) and (D) hold.

Conversely, suppose that  $S$  satisfies Conditions (A)-(D). Our aim now is to construct a semigroup  $Q$  in which if  $S$  is embedded as a left I-order in  $Q$ . We remark that from (C),  $\lambda$  is an equivalence and from the definition of  $\lambda$ ,  $\{0\}$  is a  $\lambda$ -class. Let

$$\Sigma = \{(a, b) \in S \times S : a \mathcal{R}^* b\},$$

and

$$\Sigma^* = \{(a, b) \in \Sigma : a, b \neq 0\}.$$

On  $\Sigma$  define  $\sim$  as follows;

$(a, b) \sim (c, d) \iff a = b = c = d = 0$ , or there exist  $x, y \in S^*$  such that

$$xa = yc \neq 0, \quad xb = yd \neq 0.$$

**Lemma 3.2.**  *$\sim$  is an equivalence.*

*Proof.* It is clear that  $\sim$  is symmetric. If  $(a, b) \in \Sigma^*$ , by (D) there exist  $h \in S$  such that  $ha \neq 0$  and hence  $a \mathcal{R}^* b$ ,  $hb \neq 0$  and so that  $\sim$  is reflexive. Let

$$(a, b) \sim (c, d) \sim (p, q),$$

where  $(a, b), (c, d)$  and  $(p, q)$  in  $\Sigma^*$ . Then there exist  $x, y, \bar{x}, \bar{y}$  such that

$$xa = yc \neq 0, \quad xb = yd \neq 0 \quad \text{and} \quad \bar{x}c = \bar{y}p \neq 0, \quad \bar{x}d = \bar{y}q \neq 0.$$

To show that  $\sim$  is transitive, we have to show that, there are  $z, \bar{z} \in S$  such that  $za = \bar{z}p \neq 0, zb = \bar{z}q \neq 0$ .

Now,  $yc \lambda \bar{x}c$ . For  $Sc \neq 0$  and  $Syc \neq 0$  and clearly  $Sc \cap Syc \neq 0$ , so that  $c \lambda yc$ . Similarly,  $\bar{x}c \lambda c$ ; since  $\lambda$  is transitive, we obtain  $yc \lambda \bar{x}c$ . Hence  $wyc = \bar{w}\bar{x}c \neq 0$ . Thus  $wxa = wyc = \bar{w}\bar{x}c = \bar{w}\bar{y}p \neq 0$ , that is,  $wxa = \bar{w}\bar{y}p \neq 0$ . As  $c \mathcal{R}^* d$  we have that  $wyd = \bar{w}\bar{x}d \neq 0$  so that similarly,  $wxb = \bar{w}\bar{y}q \neq 0$  as required.  $\square$

Let  $[a, b]$  denote the  $\sim$ -equivalence class of  $(a, b)$ . We stress that  $[0, 0]$  contains only the pair  $(0, 0)$ . On  $Q = \Sigma / \sim$  define a multiplication as:

$$[a, b][c, d] = \begin{cases} [xa, yd] & \text{if } b \lambda c \text{ and } xb = yc \neq 0 \\ 0 & \text{else} \end{cases}$$

and  $0[a, b] = [a, b]0 = 00 = 0$ , where  $0 = [0, 0]$ . We put  $Q^* = Q \setminus \{[0, 0]\}$

Before we show that the above multiplication is well-defined we can notice easily that  $[xa, yd] \in Q$ . For  $xa \mathcal{R}^* xb = yc \mathcal{R}^* yd$ .

**Lemma 3.3.** *The multiplication is well-defined.*

*Proof.* Suppose that  $[a_1, b_1] = [a_2, b_2]$ ,  $[c_1, d_1] = [c_2, d_2]$  are in  $Q^*$ . Then there are elements  $x_1, x_2, y_1, y_2$  in  $S$  such that

$$\begin{aligned} x_1 a_1 &= x_2 a_2 \neq 0, \\ x_1 b_1 &= x_2 b_2 \neq 0, \\ y_1 c_1 &= y_2 c_2 \neq 0, \\ y_1 d_1 &= y_2 d_2 \neq 0. \end{aligned}$$

Now,

$$[a_1, b_1][c_1, d_1] = \begin{cases} [wa_1, \bar{w}d_1] & \text{if } b_1 \lambda c_1 \text{ and } wb_1 = \bar{w}c_1 \neq 0 \\ 0 & \text{else} \end{cases}$$

and

$$[a_2, b_2][c_2, d_2] = \begin{cases} [za_2, \bar{z}d_2] & \text{if } b_2 \lambda c_2 \text{ and } zb_2 = \bar{z}c_2 \neq 0 \\ 0 & \text{else.} \end{cases}$$

Notice that as  $b_1 \lambda b_2$  and  $c_1 \lambda c_2$ , we have that  $b_1 \lambda c_1$  if and only if  $b_2 \lambda c_2$ . Hence  $[a_1, b_1][c_1, d_1] = 0$  if and only if  $[a_2, b_2][c_2, d_2] = 0$ . We now assume that  $b_1 \lambda c_1$  (and so  $b_2 \lambda c_2$  also).

We have to prove that  $[wa_1, \bar{w}d_1] = [za_2, \bar{z}d_2]$  that is,

$$xwa_1 = yza_2 \neq 0, \quad x\bar{w}d_1 = y\bar{z}d_2 \neq 0, \quad \text{for some } x, y \in S.$$

Since  $wb_1 = \bar{w}c_1 \neq 0$ ,  $zb_2 = \bar{z}c_2 \neq 0$  and  $a_1 \mathcal{R}^* b_1, a_2 \mathcal{R}^* b_1$  we have  $wa_1 \neq 0$  and  $za_2 \neq 0$ . Hence

$$wa_1 \lambda a_1 \lambda x_1 a_1 = x_2 a_2 \lambda a_2 \lambda za_2.$$

By (C)  $wa_1 \lambda za_2$ , that is,  $xwa_1 = yza_2 \neq 0$  for some  $x, y \in S$ .

The following lemma is essentially Lemma 4.8 in [7]. We give it for completeness.

**Lemma 3.4.** *Let  $a, b, c, d, s, t, x, y$  be non-zero elements of  $S$  which satisfy*

$$sa = tc \neq 0, \quad sb = td \neq 0, \quad xa = yc \neq 0.$$

*Then  $xb = yd \neq 0$ .*

*Proof.* Since  $sa \neq 0, xa \neq 0$  and  $sa \lambda a \lambda xa$ , then  $sa \lambda xa$  that is, there are elements  $w, z \in S$  such that  $zsa = wxa \neq 0$ . Since  $S$  is 0-cancellative and  $a \neq 0$  we have  $zs = wx \neq 0$ . Thus by categoricity at 0

$$ztc = zsa = wxa = wyc \neq 0.$$

Cancelling  $c$  gives  $zt = wy \neq 0$ . By categoricity at 0 again we have

$$wxb = zsb = ztd = wyd \neq 0.$$

Hence  $xb = yd \neq 0$ . □

We can apply it as follows: since  $x_1a_1 = x_2a_2 \neq 0, x_1b_1 = x_1b_2 \neq 0$  and  $xwa_1 = yza_2$ , then  $xwb_1 = yzb_2 \neq 0$ . Now,  $wb_1 = \bar{w}c_1, zb_2 = \bar{z}c_2$ , then

$$x\bar{w}c_1 = xwb_1 = yzb_2 = y\bar{z}c_2 \neq 0.$$

Reapply the same lemma to get  $x\bar{w}d_1 = y\bar{z}d_2 \neq 0$ . □

**Lemma 3.5.** *The multiplication is associative.*

*Proof.* Let  $[a, b], [c, d], [p, q] \in Q^*$  and set

$$X = ([a, b][c, d])[p, q] = \begin{cases} [xa, yd][p, q] & \text{if } b \lambda c \text{ and } xb = yc \neq 0 \\ 0 & \text{else} \end{cases}$$

and

$$Y = [a, b]([c, d][p, q]) = \begin{cases} [a, b][\bar{x}c, \bar{y}q] & \text{if } d \lambda p \text{ and } \bar{x}d = \bar{y}p \neq 0 \\ 0 & \text{else.} \end{cases}$$

Suppose that  $X = 0$ . If  $b \not\lambda c$ , then either  $d \not\lambda p$  (in which case  $Y = 0$ ) or,  $d \lambda p$  and  $\bar{x}d = \bar{y}p \neq 0$ , for some  $\bar{x}, \bar{y} \in S$ . Then  $\bar{x}c \neq 0$  and as  $c \lambda \bar{x}c, b \not\lambda \bar{x}c$ , so that again,  $Y = 0$ .

On the other hand, if  $b \lambda c$  so that  $xb = yc \neq 0$  for some  $x, y \in S$ , and if  $yd \not\lambda p$ , then  $d \not\lambda p$  so that  $Y = 0$ .

Conversely, if  $Y = 0$ , then if  $d \not\lambda p$  we have either  $b \not\lambda c$  (which case  $X = 0$ ) or  $b \lambda c$  and  $yd \neq 0$ . In this case,  $p \not\lambda yd$ , so that  $X = 0$ . If  $d \lambda p$ , then we must have that that  $b \not\lambda \bar{x}c$ , so that  $b \not\lambda c$  and again  $X = 0$ . We therefore assume that  $X \neq 0$  and  $Y \neq 0$ . Then

$$X = [xa, yd][p, q] = [sxa, rq], syd = rp \neq 0$$

$$Y = [a, b][\bar{x}c, \bar{y}q] = [\bar{s}a, \bar{r}\bar{y}q], \bar{s}b = \bar{r}\bar{x}c \neq 0.$$

for some  $s, \bar{s} \in S$ .

We have to show that  $X = Y$  i.e.

$$wsxa = \bar{w}\bar{s}a \neq 0, wrq = \bar{w}\bar{r}\bar{y}q \neq 0$$

for some  $w, \bar{w} \in S$ . By 0-cancellativity this equivalent to  $wsx = \bar{w}\bar{s} \neq 0, wr = \bar{w}\bar{r}\bar{y} \neq 0$ . Since  $xb \neq 0, sx \neq 0$  and  $S$  categorical at 0 we have  $sxb \neq 0$  also,  $\bar{s}\bar{b} \neq 0$ . Hence  $sxb \lambda \bar{s}\bar{b}$ , and so there exist  $w, \bar{w} \in S$  such that  $wsxb = \bar{w}\bar{s}\bar{b} \neq 0$ . As  $S$  0-cancellative, we have  $wsx = \bar{w}\bar{s} \neq 0$ .

Now, since  $wsxb = \bar{w}\bar{s}\bar{b} \neq 0$  and  $\bar{s}\bar{b} = \bar{r}\bar{x}\bar{c} \neq 0, xb = yc \neq 0$  we have  $wsyc = \bar{w}\bar{r}\bar{x}\bar{c} \neq 0$ . As  $S$  is 0-cancellative we have  $wsy = \bar{w}\bar{r}\bar{x} \neq 0$ , then  $wsyd = \bar{w}\bar{r}\bar{x}\bar{d} \neq 0$ , but  $syd = rp \neq 0$  and  $\bar{x}\bar{d} = \bar{y}\bar{p} \neq 0$  so that  $wrp = \bar{w}\bar{r}\bar{y}\bar{p} \neq 0$ . Thus  $wr = \bar{w}\bar{r}\bar{y} \neq 0$  as required.  $\square$

Let  $[a, b] \in Q^*$ , by (D) for  $a \in S^*$  there exist  $x \in S$  such that  $xa \neq 0$ . Clearly,  $[xa, xb] \in Q^*$ . There exist  $t \in S$  with  $txa \neq 0$ , so  $(tx)a = t(xa) \neq 0$  and as  $a \mathcal{R}^* b$ ,  $(tx)b = t(xb) \neq 0$ . Then the following lemma is clear.

**Lemma 3.6.** *If  $[a, b], [xa, xb] \in Q^*$ , then  $[a, b] = [xa, xb]$ .*

If  $a \in S^*$ , by (D) there exist  $x \in S$  such that  $xa \neq 0$ . From Lemma 2.2, we get  $x \mathcal{R}^* xa$ . Hence  $[x, xa] \in Q^*$ . If  $(y, ya) \in \Sigma^*$ , then as  $xa \lambda ya$  there exist  $s, \acute{s} \in S$  with  $sxa = \acute{s}ya \neq 0$  and we have  $sx = \acute{s}y \neq 0$ , that is,  $[x, xa] = [y, ya]$ . Hence we have got the first part of the following lemma:

**Lemma 3.7.** *The mapping  $\theta : S \rightarrow Q$  defined by  $0\theta = 0$  and for  $a \in S^*$ ,  $a\theta = [x, xa]$  where  $x \in S^*$  with  $xa \neq 0$ , is an embedding of  $S$  into  $Q$ .*

*Proof.* If  $a, b \neq 0$  and  $a\theta = b\theta$ , that is,  $[x, xa] = [y, yb]$  for some  $x, y \in S$  and  $xa \neq 0, yb \neq 0$ , then there exist  $w, \bar{w} \in S$  such that

$$wx = \bar{w}y \neq 0, wxa = \bar{w}yb \neq 0.$$

Then,  $wxa = wxb \neq 0$  and as  $S$  is 0-cancellative we have  $a = b$ . Thus  $\theta$  is one-one.

To show that  $\theta$  is a homomorphism, let  $a, b \in S^*$  and  $a\theta = [s, sa], b\theta = [t, tb]$  where  $sa \neq 0$  and  $tb \neq 0$ .

Suppose that  $ab = 0$ . If  $sa \lambda t$ , then  $usa = vt \neq 0$  for some  $u, v \in S$ . By categoricity at 0,  $usab = vtb \neq 0$ , a contradiction. Hence  $sa \not\lambda t$  and  $a\theta b\theta = 0 = (ab)\theta$ .

Assume therefore that  $ab \neq 0$ . Let  $(ab)\theta = [x, xab]$  where  $xab \neq 0$ . By categoricity at 0,  $sab \neq 0$ . Hence  $sab \lambda b \lambda tb$ , so that  $wsab = \bar{w}tb \neq 0$ .

Since  $S$  is 0-cancellative  $wsa = \bar{w}t \neq 0$ , that is,  $sa \lambda t$  and we have  $a\theta b\theta \neq 0$ . Moreover from  $xa \neq 0$  and  $sa \neq 0$  we have  $sa \lambda xa$ , that is, there exist  $m, n \in S$  such that  $msa = nxa \neq 0$ . By cancelling  $a$  we have  $ms = nx \neq 0$  and by



categoricity at 0,  $msab = nxab \neq 0$ . Thus

$$\begin{aligned}
 a\theta b\theta &= [s, sa][t, tb] \\
 &= [ws, \bar{w}tb] \\
 &= [ws, wsab] \\
 &= [s, sab] \quad \text{by Lemma 3.6} \\
 &= [x, xab] \\
 &= (ab)\theta.
 \end{aligned}$$

□

**Lemma 3.8.** *The semigroup  $Q$  is regular.*

*Proof.* Let  $[a, b] \in Q^*$ , then since  $[b, a] \in Q^*$  we get

$$\begin{aligned}
 [a, b][b, a][a, b] &= [xa, xa][a, b] \quad \text{for some } x \in S \text{ with } xb \neq 0 \\
 &= [a, a][a, b] \quad \text{by Lemma 3.6} \\
 &= [ya, yb] \quad \text{for some } y \in S \text{ with } ya \neq 0 \\
 &= [a, b] \quad \text{by Lemma 3.6 .}
 \end{aligned}$$

□

For any  $a \in S^*$  we have  $[a, a] \in Q^*$  and  $[a, a][a, a] = [xa, xa] = [a, a]$  by Lemma 3.6, that is,  $[a, a]$  is idempotent. The next lemma describe the form of  $E(Q)$ .

**Lemma 3.9.**  *$E(Q) = \{[a, a], a \in S^*\} \cup \{0\}$  and forms a semilattice.*

*Proof.* Let  $[a, b] \in E(Q^*)$ , then  $[a, b][a, b] = [a, b]$ . Hence  $[xa, yb] = [a, b]$  where  $xb = ya \neq 0$ . Thus there exists  $t, r \in S^*$  such that  $txa = ra \neq 0, tyb = rb \neq 0$ . By (B),  $tx = ty = r \neq 0$ , then  $x = y$  and  $a = b$ .

For  $[a, a], [b, b] \in E(Q^*)$  we have  $[b, b][a, a] = 0 \iff a \not\wedge b \iff [a, a][b, b] = 0$  and if  $a \wedge b$ , then

$$\begin{aligned}
 [a, a][b, b] &= [\acute{s}a, \acute{t}b] \quad \text{where } \acute{s}a = \acute{t}b \neq 0 \text{ for some } \acute{s}, \acute{t} \in S \\
 &= [\acute{t}b, \acute{s}a] \quad \text{where } \acute{s}a = \acute{t}b \neq 0 \\
 &= [b, b][a, a].
 \end{aligned}$$

□

We can note easily that  $[b, a]$  is unique, which means that it is the inverse of  $[a, b]$ .

**Lemma 3.10.** *The semigroup  $Q$  is primitive.*

*Proof.* Suppose that  $[a, a], [b, b] \in E(Q^*)$  are such that  $[a, a] \leq [b, b]$ . Then

$$\begin{aligned} [a, a] &= [a, a][b, b] \\ &= [xa, yb] \quad \text{for some } x, y \in S \text{ where } xa = yb \\ &= [yb, yb] \\ &= [b, b] \quad \text{by Lemma 3.6.} \end{aligned}$$

□

By Lemma 3.7 we can regard  $S$  as a subsemigroup of  $Q$ . Let  $[a, b] \in Q^*$  and  $a\theta = [x, xa], b\theta = [y, yb]$  where  $xa \neq 0$  and  $yb \neq 0$ . Hence

$$\begin{aligned} (a\theta)^{-1}(b\theta) &= [x, xa]^{-1}[y, yb] \\ &= [xa, x][y, yb] \\ &= [txa, ryb] \quad \text{for some } t, r \in S \text{ where } tx = ry \neq 0 \\ &= [txa, txb] \\ &= [a, b] \quad \text{by Lemma 3.6.} \end{aligned}$$

Hence  $S$  is a left I-order in  $Q$ . □

It is worth pointing out that if  $e \in E(Q^*)$ , then  $e = a^{-1}a$  for some  $a \in S^*$ . For  $e = a^{-1}b \in E(Q^*)$  as  $a \mathcal{R} b$  we have  $b = ae$  and  $a = be$ . Then it is clear that  $a = b$ .

A *Brandt semigroup* is a completely 0-simple inverse semigroup. By Theorem II.3.5 in [8] every Brandt semigroup is isomorphic to  $B(G, I)$  for some group  $G$  and non-empty set  $I$  where  $B(G, I)$  is constructed as follows:

As a set  $B(G, I) = (I \times G \times I) \cup \{0\}$ , the binary operation is defined by

$$(i, a, j)(k, b, l) = \begin{cases} (i, ab, l) & \text{if } j = k \\ 0 & \text{else} \end{cases}$$

$$(i, a, j)0 = 0(i, a, j) = 00 = 0.$$

Note that every Brandt semigroup is a primitive inverse semigroup.

**Example 3.1.** [9] Let  $H$  be a left order in a group  $G$ , and let  $\mathcal{B}^0 = \mathcal{B}^0(G, I)$  be a Brandt semigroup over  $G$  where  $|I| \geq 2$ . Fix  $i \in I$  and let

$$S_i = \{(i, h, j) : h \in H, j \in I\} \cup \{0\}.$$

Then  $S_i$  is a straight left I-order in  $\mathcal{B}^0$ .

To see this, notice that  $S_i$  is a subsemigroup,  $0 = 0^{-1}0$ , and for any  $(j, g, k) \in \mathcal{B}^0$ , we may write  $g = a^{-1}b$  where  $a, b \in H$  and then

$$(j, g, k) = (i, a, j)^{-1}(i, b, k)$$

where  $(i, a, j), (i, b, k) \in S_i$ .

Again, it is easy to see that  $S_i$  is not a left order in  $\mathcal{B}^0$ .

Let  $\{S_i : i \in I\}$  be a family of disjoint semigroups with zero, and put  $S_i^* = S_i \setminus \{0\}$ . Let  $S = \bigcup_{i \in I} S_i^* \cup 0$  with the multiplication

$$a * b = \begin{cases} ab & \text{if } a, b \in S_i \text{ for some } i \text{ and } ab \neq 0 \text{ in } S_i \\ 0 & \text{else.} \end{cases}$$

With this multiplication  $S$  is a semigroup called *0-direct union* of the  $S_i$ . In [2], it is shown that every primitive inverse semigroup with zero is a 0-direct union of Brandt semigroups.

**Corollary 3.11.** *A semigroup  $S$  is a left I-order in a Brandt semigroup  $Q$  if and only if  $S$  satisfies that conditions in Theorem 3.1 and in addition, for all  $a, b \in S^*$  there exist  $c, d \in S$  such that  $ca \mathcal{R}^* d \lambda b$ .*

*Proof.* Suppose that  $S$  is a left I-order in  $Q$  and let  $a, b \in S^*$ . Since  $Q$  has a single non-zero  $\mathcal{D}$ -class, there exists  $q \in Q$  such that  $a \mathcal{R} q \mathcal{L} b$  in  $Q$ . Let  $q = c^{-1}d$  where  $c, d \in S$  and  $c \mathcal{R} d$ . Then  $ca \mathcal{R} d$  and  $d \mathcal{L} c^{-1}d \mathcal{L} b$  that is,  $ca \mathcal{R} d \mathcal{L} b$ . By Proposition 2.4  $ca \mathcal{R}^* d \lambda b$ .

On the other hand, if  $S$  satisfies the given conditions, then we can show that  $Q$  is Brandt. For, if  $q = a^{-1}b, p = c^{-1}d \in Q^*$ , then  $b, d \in S^*$  so there exist  $u, v \in S$  with  $ub \mathcal{R}^* v \lambda d$ . In  $Q$ ,  $ub \mathcal{R} v \mathcal{L} d$ , so that  $b$  and  $d$  (and hence  $q$  and  $p$ ) lie in the same Brandt subsemigroup of  $Q$ .  $\square$

**Lemma 3.12.** *Let  $Q = \bigcup_{i \in I} Q_i$  be a primitive inverse semigroup where  $Q_i$  is a Brandt semigroup. If  $S$  is a left I-order in  $Q$ , then  $S$  is a 0-direct union of semigroups that are left I-orders in the Brandt semigroups  $Q_i$ 's.*

#### 4. UNIQUENESS

In this section we show that a semigroup  $S$  has, up to isomorphism, at most one primitive inverse semigroup of left I-quotients.

**Definition 4.1.** [9] Let  $S$  be a subsemigroup of  $Q$  and let  $\phi : S \rightarrow P$  be a morphism from  $S$  to a semigroup  $P$ . If there is a morphism  $\bar{\phi} : Q \rightarrow P$  such that  $\bar{\phi}|_S = \phi$ , then we say that  $\phi$  *lifts to*  $Q$ . If  $\phi$  lifts to an isomorphism, then we say that  $Q$  and  $P$  are *isomorphic over*  $S$ .

On a straight left I-order semigroup  $S$  in a semigroup  $Q$  we define an ternary relation  $\mathcal{T}^Q$  from [9] on  $S$  as follows:

$$(a, b, c) \in \mathcal{T}^Q \iff ab^{-1}Q \subseteq c^{-1}Q.$$

Since every left I-order in a primitive inverse semigroup is straight, then we are able to use the following result.

**Corollary 4.2.** [9] *Let  $S$  be a straight left I-order in  $Q$  and let  $\phi : S \rightarrow P$  be an embedding of  $S$  into an inverse semigroup  $P$  such that  $S\phi$  is a straight left I-order in  $P$ . Then  $Q$  is isomorphic to  $P$  over  $S$  if and only if for any  $a, b, c \in S$ :*  
(i)  $(a, b) \in \mathcal{R}_S^Q \Leftrightarrow (a\phi, b\phi) \in \mathcal{R}_{S\phi}^P$ ; and  
(ii)  $(a, b, c) \in \mathcal{T}_S^Q \Leftrightarrow (a\phi, b\phi, c\phi) \in \mathcal{T}_{S\phi}^P$ .

**Theorem 4.3.** *Let  $S$  be a left I-order in a primitive inverse semigroup  $Q$ . If  $\phi : S \rightarrow T$  is an isomorphism where  $T$  is a left I-order in a primitive inverse semigroup  $P$ , then  $\phi$  lifts to an isomorphism  $\bar{\phi} : Q \rightarrow P$*

*Proof.* Let  $\phi : S \rightarrow T$  be as given. From Proposition 2.4  $S$  and  $T$  both contain 0 and clearly  $\phi$  preserves this. Let  $a, b \in S^*$  with  $a \mathcal{R} b$  in  $Q$ . By Condition (D) there exists  $c \in S$  with  $ca \neq 0$  and hence  $cb \neq 0$ . It follows that  $(c\phi)(a\phi), (c\phi)(b\phi)$  are non-zero in  $P$ , so that  $a\phi \mathcal{R} b\phi$  in  $P$ . Also  $\phi$  preserves  $\mathcal{L}$ . If  $a, b \in S^*$  and  $a \mathcal{L} b$  in  $Q$ , then  $a \lambda b$  in  $S$  so that  $ca = db \neq 0$  for some  $c, d \in S$ . Then  $c\phi a\phi = d\phi b\phi \neq 0$  so that  $a\phi \mathcal{L} b\phi$  in  $P$ .

We now show that  $\phi$  preserves  $\mathcal{T}_S^Q$ . Suppose therefore that  $a, b, c \in S$  and  $ab^{-1}Q \subseteq c^{-1}Q$ . Then either  $ab^{-1} = 0$ , or  $ab^{-1} \mathcal{R} c^{-1}$  in  $Q$ . In the former case, either  $a$  or  $b$  is 0 or  $a$  and  $b$  are not  $\mathcal{L}$ -related in  $Q$  it follows that either  $a\phi$  or  $b\phi$  is 0 or  $a\phi$  and  $b\phi$  are not  $\mathcal{L}$ -related in  $P$ , giving  $(a\phi)(b\phi)^{-1} = 0$  and so  $(a\phi)(b\phi)^{-1}P \subseteq (c\phi)^{-1}P$ . On the other hand, if  $ab^{-1} \neq 0$ , then we have  $a, b \neq 0$ ,  $a \mathcal{L} b$  and  $a \mathcal{R} c^{-1}$  in  $Q$ . It follows that  $ca \neq 0$   $a\phi \mathcal{L} b\phi$  and  $(c\phi)(a\phi) \neq 0$  in  $P$ . Consequently,

$$0 \neq (a\phi)(b\phi)^{-1}P = (a\phi)P = (c\phi)^{-1}P.$$

Since  $\phi$  (and, dually,  $\phi^{-1}$ ) preserve both  $\mathcal{R}$  and  $\mathcal{T}$ , it follows from Corollary 4.2 that  $\phi$  lifts to an isomorphism  $\bar{\phi} : Q \rightarrow P$ . □

The following corollary may be deduced from the previous theorem.

**Corollary 4.4.** *If  $Q_1, Q_2$  are primitive inverse semigroups of left I-quotients of a semigroup  $S$ , then  $Q_1, Q_2$  are isomorphic by an isomorphism which restricts to the identity map on  $S$ .*

**Proposition 4.5.** *If a semigroup  $S$  has a primitive inverse semigroup  $Q$  of left I-quotients and a primitive inverse semigroup  $\acute{Q}$  of right I-quotients, then  $Q$  and  $\acute{Q}$  are both semigroups of I-quotients of  $S$ , so that  $Q \cong \acute{Q}$  by Corollary 4.4.*

*Proof.* We show that  $Q$  is a semigroup of right I-quotients of  $S$ . Let  $q \in \acute{Q}$ . If  $q = 0$ , then  $q = 0\acute{0} = 0\acute{0}$ . If  $q \in Q^*$ , then  $q = a^{-1}b$  for some  $a, b \in S^*$  where  $a \mathcal{R} b$  in  $Q$ . Then  $a \mathcal{R}^* b$  in  $S$ . Pick  $c \in S$  with  $ca \neq 0$ . Then  $ca \mathcal{R}^* cb$  in  $\acute{Q}$ , so that by the dual of Proposition 2.4  $ca \rho cb$  in  $S$ , that is  $cax = cby \neq 0$  for some  $x, y \in S$ . As  $S$  is 0-cancellative,  $ax = by \neq 0$ , so that  $xy^{-1} = a^{-1}b$  in  $Q$  (and  $\acute{Q}$ ). Thus  $S$  an I-order in  $Q$  and similarly, in  $\acute{Q}$ . □

## 5. PRIMITIVE INVERSE SEMIGROUPS OF I-QUOTIENTS

In this section we study the case where a semigroup is both a left and right I-order in a primitive inverse semigroup, that is, an I-order.

**Lemma 5.1.** *Let  $S$  have a primitive inverse semigroup  $Q$  of I-quotients. Then*

- 1)  $\mathcal{R}^* = \mathcal{R} \cap (S \times S) = \rho$ ,
- 2)  $\mathcal{L}^* = \mathcal{L} \cap (S \times S) = \lambda$ ,
- 3)  $\mathcal{H}^* = \mathcal{H} \cap (S \times S) = \tau$ .

*Proof.* (1) By Proposition 2.4  $\mathcal{R} \cap (S \times S) = \mathcal{R}^*$ . By the dual of Proposition 2.4,  $\mathcal{R} \cap (S \times S) = \rho$ . Hence  $\rho = \mathcal{R}^*$ .

2) Similar.

3) Immediate from 1) and 2).  $\square$

Since  $\mathcal{H}$  is a congruence on any primitive inverse semigroup the following corollary is clear.

**Corollary 5.2.** *Let  $Q$  be primitive inverse semigroup of I-quotients of a semigroup  $S$ . Then  $\mathcal{H}^*$  is a congruence relation on  $S$ .*

If  $S$  is an I-order in a primitive inverse semigroup  $Q$ , then  $S$  satisfies the conditions in Theorem 3.1 and in addition, the duals  $(\acute{C})$  and  $(\acute{D})$  of (C) and (D). We can reduce these conditions by using the next lemma.

**Lemma 5.3.** *Left  $S$  be a left I-order in a primitive inverse semigroup  $Q$  and suppose that  $aS \neq 0$  for all  $a \in S^*$ . Then*

$$\mathcal{R}^* \subseteq \rho \text{ if and only if } S \text{ is an I-order in } Q.$$

*Proof.* Suppose that  $\mathcal{R}^* \subseteq \rho$ , by Proposition 2.4,  $\mathcal{R}^* = \rho$ , and so  $\rho$  is transitive. By dual of Theorem 3.1  $S$  is a right I-order in a primitive inverse semigroup  $\acute{Q}$ . By Proposition 4.5  $Q \cong \acute{Q}$  and  $S$  is an I-order in  $Q$ . On the other hand, if  $S$  is an I-order in  $Q$ , then by Lemma 5.1,  $\mathcal{R}^* = \rho$  as required.  $\square$

Now we introduce condition (E) which appeared in [7] for a semigroup with zero as follows:

(E)  $a \rho b$  if and only if  $a = b = 0$  or there exist an element  $x$  in  $S$  such that

$$xa \neq 0 \text{ and } xb \neq 0.$$

**Lemma 5.4.** *For a semigroup  $S$ , the following conditions are equivalent:*

- (1)  $S$  has a primitive inverse semigroup of I-quotients;
- (2)  $S$  is 0-cancellative, categorical at 0, and  $S$  satisfies (D),  $(\acute{D})$ , (E) and  $(\acute{E})$ .

*Proof.* If (1) holds, then by Theorem 3.1 and its dual, we need only to show that  $S$  satisfies (E) and its dual. Suppose that  $a \rho b$  and  $a, b \neq 0$ . By Lemma 5.1  $a \mathcal{R}^* b$ ,

and by (D) there is  $x \in S$  such that  $xa \neq 0$ , and so  $xb \neq 0$ . Conversely, if  $xa \neq 0$  and  $xb \neq 0$ , then using (D),  $xa \rho x$  and  $x \rho xb$ . Since  $\rho$  is transitive,  $xa \rho xb$ , that is,  $xat = xbr \neq 0$  for some  $t, r \in S$ , by cancelling  $x$  we have  $at = br \neq 0$ . Thus  $a \rho b$ . Similarly  $S$  satisfies  $(\acute{E})$ .

If (2) holds, we show that  $\lambda$  and  $\rho$  are transitive. In order to prove this, we need to show that  $\mathcal{R}^* = \rho$  and  $\mathcal{L}^* = \lambda$ . Let  $a \mathcal{R}^* b$ , then either  $a = b = 0$  or  $a, b \neq 0$  and by (D),  $xa \neq 0$  for some  $x \in S$  and as  $a \mathcal{R}^* b$ , then  $xb \neq 0$ . By (E) we have that  $a \rho b$  so that  $\mathcal{R}^* \subseteq \rho$ .

Conversely, if  $a \rho b$ , then either  $a = b = 0$  (so that  $a \mathcal{R}^* b$ ) or  $ah = bk \neq 0$  for some  $h, k \in S$ . Suppose now that  $u, v \in S^1$  and  $ua = va$ . If  $ua = va \neq 0$ , then by categoricity at 0,  $uah = vah \neq 0$ , so that  $ubk = vbk \neq 0$  and 0-cancellativity gives  $ub = vb \neq 0$ . On the other hand, if  $ua = va = 0$ , then  $u, v \in S$  and  $ubk = vbk = 0$ . By categoricity at 0,  $ub = vb = 0$ . Similarly  $ub = vb$  implies  $ua = va$ . Hence  $a \mathcal{R}^* b$ .  $\square$

We now summaries the result of this Section.

**Proposition 5.5.** *For a semigroup  $S$ , the following conditions are equivalent:*

- (1)  *$S$  is an I-order in a primitive inverse semigroup;*
- (2)  *$S$  satisfies conditions (A), (B), (C), (D),  $(\acute{C})$  and  $(\acute{D})$ ;*
- (3)  *$S$  satisfies conditions (A), (B), (C), (D),  $(\acute{D})$  and  $\mathcal{R}^* \subseteq \rho$ ;*
- (4)  *$S$  satisfies conditions (A), (B), (D),  $(\acute{D})$ , (E) and  $(\acute{E})$ .*

*Proof.* The equivalence of (1) and (2) follows from Theorem 3.1 and its dual, and Proposition 4.5. The equivalence of (1) and (3) is immediate from Theorem 3.1 and its dual, and Lemma 5.1 and 5.3. Finally, the equivalence of (1) and (4) is gives by Lemma 5.4.  $\square$

## 6. THE ABUNDANT CASE

A semigroup  $S$  is a *left abundant* if each  $\mathcal{R}^*$ -class contains at least one idempotent. Dually, we can define a *right abundant* semigroup. A semigroup is *abundant* if it is both left and right abundant. If  $S$  is left (right) abundant and  $E(S)$  is a semilattice, then  $S$  is *left (right) adequate*. Note that in this case, the idempotent in the  $\mathcal{R}^*$ -class ( $\mathcal{L}^*$ -class) of  $a$  is unique. We denote it by  $a^+$  ( $a^*$ ). A semigroup which is both left and right adequate will be called an *adequate semigroup*. Fountain [4] has generalised the Rees theorem to show that every abundant semigroup in which the non-zero idempotents primitive, is isomorphic to what he calls a *PA-blocked Rees matrix semigroup*. We refer the interested reader to [4] for more details. It is clear that if an abundant semigroup is a left I-order in a primitive inverse semigroup, then it is adequate. More than this, it must be

ample, as we now explain.

We recall that a semigroup  $S$  is a left (right) ample if and only if  $S$  is left (right) adequate and satisfies the left (right) ample condition which is;

$$(ae)^+a = ae \quad (a(ea)^* = ea) \quad \text{for all } a \in S \text{ and } e \in E(S).$$

A semigroup is an ample semigroup if it is both left and right ample. From [10] a semigroup  $S$  is left ample if and only if it embeds in an inverse semigroup  $T$  such that  $\mathcal{R} \cap (S \times S) = \mathcal{R}^*$ . If a left ample semigroup  $S$  has a primitive inverse semigroup of left I-quotients  $Q$ , then for any  $a \in S$  we have  $a\mathcal{R}^*a^+$ , by Proposition 2.4  $a\mathcal{R}a^+$ , that is  $a^+ = aa^{-1}$ . Hence the following lemma is clear.

**Lemma 6.1.** *Let  $S$  be a left I-order in a primitive inverse semigroup  $Q$ . Then the following are equivalent:*

- 1)  $S$  is left adequate;
- 2)  $S$  is left ample.

In the next lemma we introduce an equivalent condition for categoricity at 0 for any primitive ample semigroup with zero.

**Lemma 6.2.** *Let  $S$  be a primitive ample semigroup with zero. Then the following are equivalent*

- (i)  $S$  is categorical at 0;
- (ii)  $a^* = b^+ \iff ab \neq 0$  for  $a$  and  $b$  in  $S^*$ .

*Proof.* (i)  $\implies$  (ii) Let  $a, b \in S$ . If  $ab \neq 0$ , then  $aa^*b^+b \neq 0$ , so  $a^*b^+ \neq 0$  and so by primitivity  $a^* = b^+$ . Conversely, if  $a^* = b^+$ , then  $aa^* \neq 0, a^*b = b^+b \neq 0$ , so by categoricity at 0,  $ab = aa^*b \neq 0$ .

(ii)  $\implies$  (i) Suppose that  $ab \neq 0, bc \neq 0$  where  $a, b, c \in S$ , then  $a^* = b^+$  and  $b^* = c^+$ . Hence  $b^+(bc) \neq 0$  gives  $b^+ = (b^+)^* = (bc)^+$ , but  $b^+ = a^*$ , that is,  $a^* = (bc)^+$ . By assumption  $abc \neq 0$ .  $\square$

We can offer some simplification of Theorem 3.1 in the case that  $S$  is adequate.

**Proposition 6.3.** [4, Proposition 5.5] *For a semigroup  $S$  with zero, the following conditions are equivalent:*

- 1)  $S$  is categorical at 0, 0-cancellative and satisfies:  
for each element  $a$  of  $S$ , there is an element  $e$  of  $S$  such that  $ea = a$  and an element  $f$  of  $S$  such that  $af = a$  .....(\*)
- 2)  $S$  is a primitive adequate semigroup.
- 3)  $S$  is isomorphic to PA-blocked Rees  $I \times I$  matrix semigroup  $\mathcal{M}(M_{\alpha\beta}; I, I, \Gamma; P)$  where the sandwich matrix  $P$  is diagonal and  $p_{ii} = e_\alpha$  for each  $i \in I_\alpha, \alpha \in \Gamma$ .

From the above lemma and Theorem 3.1, the following lemma is clear.

**Lemma 6.4.** *For a semigroup  $S$  with zero, the following conditions are equivalent:*

- 1)  $S$  is abundant and left I-order in a primitive inverse semigroup  $Q$ ;
- 2)  $S$  is a primitive adequate semigroup and  $\lambda$  is transitive;
- 3)  $S$  is 0-cancellative, categorical at 0,  $S$  satisfies (\*) and  $\lambda$  is transitive.

In the two-sided case we have the following.

**Lemma 6.5.** *For a semigroup with zero, the following conditions are equivalence:*

- 1)  $S$  is abundant and an I-order in a primitive inverse semigroup  $Q$ ;
- 2)  $S$  is primitive adequate and  $\lambda, \rho$  are transitive;
- 3)  $S$  is 0-cancellative, categorical at 0, satisfies (\*) and  $\lambda, \rho$  are transitive.

We cannot use any thing in [9], because we did not talk about zero in that paper or we should add it or rather we should cancel the following Prop.

**Proposition 6.6.** *Let  $S$  be a left ample semigroup and left I-order in a primitive inverse semigroup  $Q$ . If  $S$  is a union of  $\mathcal{R}$ -classes of  $Q$ , then  $Q \cong \Sigma(S)$ .*

*Proof.* By Lemma 4.4, it is enough to show that  $\Sigma(S)$  is primitive, since by Theorem 3.7 and Lemma 3.6 of [9],  $S$  is a left I-order in  $\Sigma(S)$ . Let  $0 \neq e \leq f$  in  $\Sigma(S)$ , then  $e = a^{-1}a$  and  $f = b^{-1}b$  for some  $a, b \in S$  where  $e, f \in E(\Sigma(S))$ . We have  $0 \neq e = ef$ , so that  $ab^{-1} \neq 0$  and  $ab^{-1} = c^{-1}d$  for some  $c, d \in S$  with  $c \mathcal{R}^* d$  in  $S$  and so  $c \mathcal{R} d$  in  $\Sigma(S)$ . Then by Lemma 2.6 of [9],  $ca = db \neq 0$ , so that in  $Q$ ,  $a \mathcal{L} b$  and so  $a \mathcal{L} b$  in  $S$  by Lemma 3.6 of [9] therefore in  $\Sigma(S)$ . Hence  $e = f$ .  $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF YORK, HESLINGTON, YORK YO10 5DD, UK

*E-mail address:* ng521@york.ac.uk